# MATH 2050 - Limit of functions

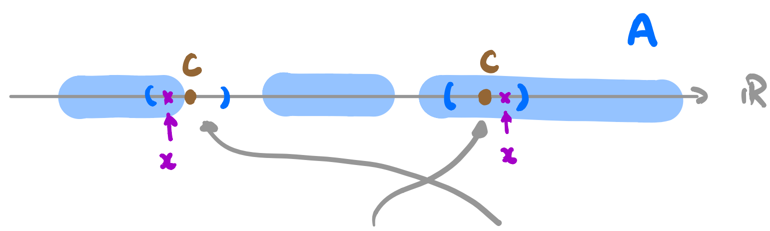
(Reference: Bartle §4.1)

**GOAL:** Define $\lim_{x\to c} f(x) = L$ for functions $f: A \subseteq \mathbb{R} \to \mathbb{R}$

We shall only define $\lim_{x\to c} f(x)$ for those "$c$"'s which are "cluster point" of $A$.

**IDEA:** $f(x) \overset{\varepsilon}{\approx} L$ when $x \overset{\delta}{\approx} c$ and $\underbrace{x \in A}_{\text{so } f(x) \text{ is defined.}}$

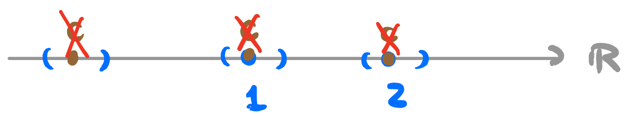
**Def$^{\underline{n}}$:** Let $A \subseteq \mathbb{R}$. We say that $c \in \mathbb{R}$ is a cluster point of $A$ iff $\forall \delta > 0$, $\exists x \in A$ s.t. $x \neq c$ and $|x - c| < \delta$



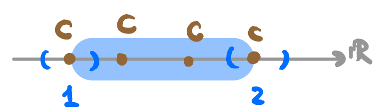

**Remark:** A cluster pt. $c \in \mathbb{R}$ may or may not belong to $A$.

**Examples:**

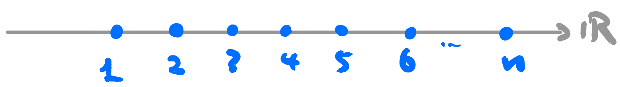
- $A = \{1, 2\}$ NO cluster pt.



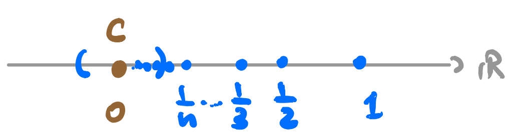

- $A = (0, 1)$ Any $c \in [0, 1]$ is a cluster pt




- $A = \{a_1, \dots, a_n\}$ NO cluster pt.
- $A = \mathbb{N}$ NO cluster pt.




- $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ ONLY 1 cluster pt $c = 0$



**Prop:** $c \in \mathbb{R}$ is a cluster point of $A$

$\iff \exists$ seq. $(a_n)$ in $A$ s.t. $a_n \neq c$ $\forall n \in \mathbb{N}$

and $\lim (a_n) = c$

**Sketch of Proof:** ($\Rightarrow$) Take $\delta_n := \frac{1}{n}$, by def$^{\underline{n}}$. $\exists$ $a_n \in A$ st

$a_n \neq c$ and $|a_n - c| < \delta_n = \frac{1}{n} \xrightarrow{\text{as } n\to\infty} 0$

$\blacksquare$

We now state the most important definition for this chapter.

**Def$^{\underline{n}}$:** Let $f : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function.

Suppose $c \in \mathbb{R}$ is a cluster point of $A$.

We say that "$f$ converges to $L \in \mathbb{R}$ at $c$", written

"$\lim\limits_{x \to c} f(x) = L$" or "$f(x) \to L$ as $x \to c$"

iff $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$|f(x) - L| < \varepsilon$, $\forall x \in A$ where $\underbrace{0 < |x - c|}_{\text{so } x \neq c} < \delta$

**Example 1**: Let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) := x$ for all $x \in \mathbb{R}$.

$$\lim_{x \to c} f(x) = c \qquad \forall\, c \in \mathbb{R}.$$

Pf: Any $c \in \mathbb{R}$ is a cluster pt. of $A = \mathbb{R}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $\delta > 0$ st $\delta = \varepsilon$

THEN, $\forall\, x \in \mathbb{R}$, and $0 < |x - c| < \delta$, we have

$$|f(x) - c| = |x - c| < \delta = \varepsilon$$

$\blacksquare$

**Remark**: $\lim_{x \to c} f(x)$ may exists with $f$ being defined at $c$.

E.g.) $f: A = (0,1) \to \mathbb{R}$ ; $f(x) := x$

$$\Rightarrow \quad \lim_{x \to 1} f(x) = 1 \notin A$$

**Example 2**: $\boxed{\lim_{x \to c} x^2 = c^2}$ i.e. $f: A = \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

Pf: Fix $c \in \mathbb{R}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

if $0 < |x - c| < \delta$, then

$$|x^2 - c^2| = |x + c| \cdot |x - c| \le (|x| + |c|) \cdot |x - c| \le (2|c| + \delta) \cdot \delta < \varepsilon.$$

$|x - c| < \delta \Rightarrow |x| < |c| + \delta$

Note: Suppose $|x - c| < 1$, then

$$|x| \le |x - c| + |c| < 1 + |c|$$

Choose $\delta := \min\left\{1, \dfrac{\varepsilon}{2(1 + 2|c|)}\right\}$

THEN, $\forall x \in A = \mathbb{R}$ with $0 < |x-c| < \delta$, we have

$$|x^2 - c^2| = |x+c| \cdot |x-c| \leq (1+2|c|)\,\delta < \varepsilon$$

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<u>Example 3:</u> $\boxed{\lim_{x \to c} \frac{1}{x} = \frac{1}{c}}$ where $c \neq 0$.

Considering $f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) := \frac{1}{x}$

<u>Note</u>: Any $c \in \mathbb{R}$ is a cluster pt of $A$

<u>Pf</u>: Let's assume $c > 0$.

Let $\varepsilon > 0$ be fixed but arbitrary.

<u>Note</u>: If $|x - c| < \frac{c}{2}$, then

$$|x| > \frac{c}{2} > 0$$

Take $\delta := \min\left\{ \frac{c}{2}, \frac{\varepsilon c^2}{2} \right\} > 0$.

Then, $\forall x \in A$ and $0 < |x-c| < \delta$

we have

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{1}{|x|} \cdot \frac{1}{|c|} \cdot |x-c| < \frac{2}{c^2} \cdot \delta \leq \varepsilon$$

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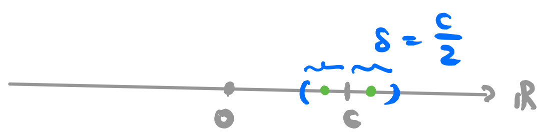
If $0 < |x-c| < \delta$, then

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x-c|}{|xc|}$$

$$= \frac{1}{|x|} \frac{1}{|c|} |x-c| < \frac{2}{c^2} \cdot \delta \leq \varepsilon$$

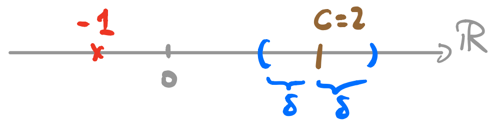
Need $|x|$ bdd away from $0$.



**Example:** $\boxed{\lim_{x \to 2} \frac{x^3-4}{x+1} = \frac{4}{3}}$

$f : A := \mathbb{R} \setminus \{-1\} \to \mathbb{R}$

$f(x) := \frac{x^3-4}{x+1}$




Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Note: If $|x-2| < 1$, then

$$1 < x < 3$$

Hence. $|x+1| > 2 > 0$

and $|3x^2+6x+8| \leq 1000$.

Choose $\delta = \min\{1, \frac{3}{500}\varepsilon\}$.

THEN, $\forall x \in A$, and $0 < |x-2| < \delta$,

$$\left| \frac{x^3-4}{x+1} - \frac{4}{3} \right| = \left| \frac{3x^3-4x-16}{3(x+1)} \right|$$

$$= \frac{|3x^2+6x+8|}{3|x+1|} \cdot |x-2|$$

$$< \frac{1000}{3 \cdot 2} \delta \leq \varepsilon$$

$\square$

Here: $0 < |x-2| < \delta$

$$\left| \frac{x^3-4}{x+1} - \frac{4}{3} \right| = \left| \frac{3(x^3-4)-4(x+1)}{3(x+1)} \right|$$

$$= \left| \frac{3x^3-4x-16}{3(x+1)} \right|$$

$$= \left| \frac{(x-2)(3x^2+6x+8)}{3(x+1)} \right|$$

$$= \frac{|3x^2+6x+8|}{3|x+1|} \cdot |x-2|$$

Small

Note: $|x-2| < 1$

$\Rightarrow \quad 1 < |x| < 3$

So $|x+1| > 2 > 0$.

and $|3x^2+6x+8|$

$$\leq 3|x|^2 + 6|x| + 8$$

$$\leq 3 \cdot 3^2 + 6 \cdot 3 + 8 \leq 1000$$

**Prop:** $\lim_{x \to c} f(x)$, if exists, is unique. (Pf: Exercise!)

**Thm:** "Sequential Criteria"

$$\lim_{x \to c} f(x) = L \iff \forall \text{ seq. } (x_n) \text{ in } A \text{ s.t. } \overset{(*)}{\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}}$$

we have $\lim (f(x_n)) = L$

**Proof:** "$\Rightarrow$" Let $(x_n)$ be a seq in $A$ s.t. $(*)$ holds

Let $\varepsilon > 0$ be fixed but arbitrary.

Since $\lim_{x \to c} f(x) = L$, $\exists\ \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad \begin{matrix} x \in A \\ 0 < |x - c| < \delta \end{matrix}$$

Since$^{(*)}$ $\lim (x_n) = c$, for the $\delta > 0$ above,

$\exists\ K = K(\delta) \in \mathbb{N}$ s.t. $0 \overset{(*)}{<} |x_n - c| < \delta \quad \forall n \geq K$

$$\Rightarrow \quad |f(x_n) - L| < \varepsilon \quad \forall n \geq K$$

"$\Leftarrow$" Suppose NOT, i.e. $\exists\ \varepsilon_0 > 0$ s.t. $\forall\ \delta > 0$,

$\exists\ x_\delta \in A$ s.t. $0 < |x_\delta - c| < \delta$

BUT: $\quad |f(x_\delta) - L| \geq \varepsilon_0$

Take $\delta = \frac{1}{n}$, then get $x_n \in A$ s.t.

$$0 < |x_n - c| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - L| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad \lim (x_n) = c \qquad \text{BUT} \quad \lim (f(x_n)) \neq L$$

$x_n \neq c \quad \forall n \in \mathbb{N}$

Contradiction! $\blacksquare$

In summary, we have

<u>Setup</u>: $f: A \subseteq \mathbb{R} \to \mathbb{R}$, $c$ : a cluster pt. of $A$ (<u>Note</u>: not nec. belong to $A$)

<u>Def$^n$</u>: $\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $x \in A$ and $0 < |x - c| < \delta$

<u>Sequential Criteria</u>

$\lim_{x \to c} f(x) = L \iff \forall$ seq. $(x_n)$ in $A$ s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$

we have $\lim (f(x_n)) = L$

limit of function

limit of seq. of real numbers

<u>Remark</u>: This is helpful, in particular, to show that the limit $\lim_{x \to c} f(x)$ DOES NOT EXIST.

Taking the negation of Sequential Criteria above, we get:

<u>Cor 1</u>: $f$ DOES NOT Converge to $L$ as $x \to c$ $\iff \exists$ seq. $(x_n)$ in $A$ s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$

BUT $(f(x_n)) \not\to L$

<u>Cor 2</u>: $f$ "DIVERGES" as $x \to c$ (i.e. $f$ DOES NOT Converge to $L$ $\forall L \in \mathbb{R}$ as $x \to c$) $\iff \exists$ seq. $(x_n)$ in $A$ s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$

BUT $(f(x_n))$ is divergent.

" Divergence Criteria "

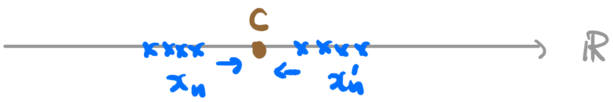
Proof of Cor. 2: "$\Leftarrow$" Easy.

"$\Rightarrow$" Argue by Contradiction. Assume $f$ diverges at $x \to c$ but the R.H.S. fails to hold.

i.e. $\forall$ seq. $(x_n)$ in $A$ st. $(*) \begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim(x_n) = c \end{cases}$

we have $\lim(f(x_n)) = L$ for some $L \in \mathbb{R}$

which may depend on the sequence $(x_n)$




Claim: The limit $L$ DOES NOT depend on $(x_n)$.

Pf of Claim: Suppose $(x_n)$, $(x_n')$ satisfy $(*)$, and

$$\lim(f(x_n)) = L \neq L' = \lim(f(x_n'))$$

Consider the new sequence

$$(y_n) := (x_1, x_1', x_2, x_2', x_3, x_3', \ldots\ldots)$$

satisfies $(*)$, then by hypothesis

$$(f(y_n)) := (f(x_1), f(x_1'), f(x_2), f(x_2'), \ldots\ldots)$$

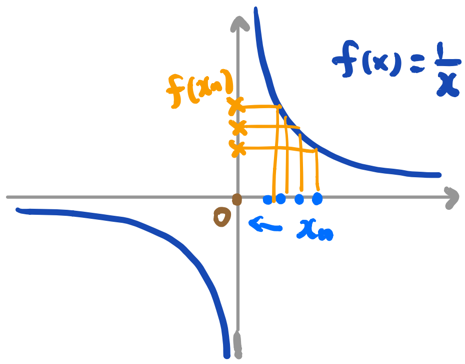
$L$ $L'$

is convergent, hence $L = L'$ $\qquad$ ______ $\square$

By sequential criteria, $\lim_{x \to c} f(x) = L$ contradiction! $\qquad$ ______ $\square$

We now look at some examples where the limit of functions does not exist.

**Example 1** : $\lim_{x \to 0} \frac{1}{x}$ DOES NOT EXIST!




$f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) := \frac{1}{x}$

**Pf**: Take $(x_n) := (\frac{1}{n})$.

Clearly, $\lim (x_n) = 0$, and

$A \ni x_n \neq 0 \quad \forall n \in \mathbb{N}$

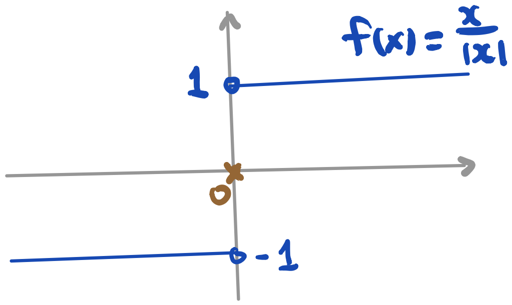
So $(*)$ is satisfied.

BUT $(f(x_n)) = (n)$ is DIVERGENT!

So we are done according to the divergence criteria above.

[ Exercise: Prove directly using $\varepsilon$-$\delta$ def$^{n}$ of limit. ]

**Example 2** : $\lim_{x \to 0} \frac{x}{|x|}$ DOES NOT EXIST.




$f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) := \frac{x}{|x|}$

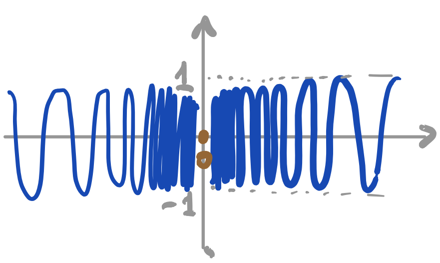
**Pf**: Take $(x_n) := \left( \frac{(-1)^n}{n} \right)^{*0} \to 0$

satisfying $(*)$, the image seq.

$(f(x_n)) = ((-1)^n)$ DIVERGENT!

$\square$

**Example 3** : $\lim_{x \to 0} \sin \frac{1}{x}$ DOES NOT EXIST!




$f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) = \sin \frac{1}{x}$

**Pf**: Take $(x_n) := \left( \frac{1}{n\pi} \right)^{*0} \to 0$ BUT $(f(x_n)) = (0)$

Take $(x_n') := \left( \frac{1}{\frac{\pi}{2} + 2n\pi} \right)^{*0} \to 0$ BUT $(f(x_n')) = (1)$

So, let $(y_n) = (x_1, x_1', x_2, x_2', x_3, x_3', \dots)^{*0} \to 0$

BUT $(f(y_n)) = (0, 1, 0, 1, 0, 1, \dots)$ DIVERGENT!

$\square$